

On the limiting characteristics for an inhomogeneous $M_t|M_t|S$ queue with catastrophes

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Abstract. We study weak ergodicity, bounds on the rate of convergence, and problems of computing of the limiting characteristics for an inhomogeneous $M_t|M_t|S$ queueing model with possible catastrophes.

1 Introduction

Qualitative and quantitative properties of inhomogeneous continuous-time Markov chains and the correspondent queueing models have been investigated since 1980's, see for instance first results in [16], [18], [19]. Queueing systems with catastrophes (queues with disasters) in different situations were studied by a number of authors, see, for instance, [8], [10, 11, 12], [22, 23].

Perturbation bounds for an inhomogeneous $M_t|M_t|N$ queue with catastrophes were obtained on [25]. First investigations for this model with catastrophes rates depending on the length of the queue were studied in [23].

It is well known that explicit expressions for the probability characteristics of stochastic models can be found only in a few special cases. If we deal with inhomogeneous Markovian model, then we must approximately calculate, in addition, the limiting probability characteristics of the process. The problem of existence and construction of limiting characteristics for time-inhomogeneous birth and death processes is important for queueing applications, see for instance, [1, 2, 3, 15, 4, 5, 6, 7, 20]. A general approach to the study of the rate of convergence for birth-death models and related bounds were considered in [19], and for finite birth-death-catastrophe models they were considered in [26]. Calculation of the limiting characteristics for the process via truncations was firstly mentioned in [17] and was considered in details in [20]. The best results in this direction for general inhomogeneous birth-death models were obtained in our recent paper [29].

Here we apply this general approach to an inhomogeneous $M_t|M_t|S$ queue with catastrophes in a general situation where catastrophe rates depend on the length of the queue. Moreover, we will obtain and discuss explicit bounds on the rate of convergence to the limiting characteristics in weak ergodic situation as well as approximation bounds of the limiting characteristics. Finally, we discuss an example of this queueing model.

Let $X = X(t)$, $t \geq 0$, be an inhomogeneous, in general, continuous-time Markov chain, which is the queue length process for the corresponding queueing model.

Let $p_{ij}(s, t) = \Pr\{X(t) = j | X(s) = i\}$, $i, j \geq 0$, $0 \leq s \leq t$ be the transition probabilities for $X = X(t)$, $p_i(t) = \Pr\{X(t) = i\}$ be its state probabilities, and $\mathbf{p}(t) = (p_0(t), p_1(t), \dots)^T$ be the corresponding probability distribution.

In the inhomogeneous case we assume that all intensity functions are linear combinations of a finite number of nonnegative functions which are locally integrable on $[0, \infty)$. Then the corresponding transposed intensity matrix is

$$A(t) = \begin{pmatrix} a_{00}(t) & \mu_1(t) + \xi_1(t) & \xi_2(t) & \xi_3(t) & \xi_4(t) & \xi_5(t) & \dots \\ \lambda_0(t) & a_{11}(t) & \mu_2(t) & 0 & 0 & 0 & \dots \\ 0 & \lambda_1(t) & a_{22}(t) & \mu_3(t) & 0 & 0 & \dots \\ 0 & 0 & \lambda_2(t) & a_{33}(t) & \mu_4(t) & 0 & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \end{pmatrix},$$

$a_{ii}(t) = -\sum_{j \neq i} a_{ji}(t)$. In addition, applying our standard approach (see details in [15, 19, 21, 20]) we suppose that the intensity matrix is essentially bounded, i. e.

$$|a_{ii}(t)| \leq L < \infty, \quad (1)$$

for almost all $t \geq 0$. Then the probabilistic dynamics of the process is represented by the forward Kolmogorov system

$$\begin{cases} \frac{dp_0}{dt} = -\lambda_0(t)p_0 + \mu_1(t)p_1 + \sum_{k \geq 1} \xi_k(t)p_k, \\ \frac{dp_k}{dt} = \lambda_{k-1}(t)p_{k-1} - (\lambda_k(t) + \mu_k(t) + \xi_k(t))p_k + \mu_{k+1}(t)p_{k+1}, k \geq 1, \end{cases} \quad (2)$$

where $\lambda_k(t) = \lambda(t)$, $\mu_k(t) = \min(k, S)\mu(t)$, and $\xi_k(t)$ are the arrival, service and catastrophe rates, respectively.

Throughout the paper by $\|\cdot\|$ we denote the l_1 -norm, i. e., $\|\mathbf{x}\| = \sum |x_i|$, and $\|B\| = \sup_j \sum_i |b_{ij}|$ for $B = (b_{ij})_{i,j=0}^\infty$. Let Ω be the set all stochastic vectors, i. e. l_1 -vectors with nonnegative coordinates and unit norm.

Then we have $\|A(t)\| = 2 \sup_k |a_{kk}(t)| \leq 2L$ for almost all $t \geq 0$. Hence, the operator function $A(t)$ from l_1 into itself is bounded for almost all $t \geq 0$ and locally integrable on $[0; \infty)$. Therefore, we can consider the forward Kolmogorov system (2)

$$\frac{d\mathbf{p}}{dt} = A(t)\mathbf{p}(t), \quad (3)$$

as a differential equation in the space l_1 with bounded operator.

It is well known (see [13]) that the Cauchy problem for differential equation (3) has a unique solution for an arbitrary initial condition, and $\mathbf{p}(s) \in \Omega$ implies $\mathbf{p}(t) \in \Omega$ for $t \geq s \geq 0$.

By $E(t, k) = E\{X(t) | X(0) = k\}$ denote the mean (the mathematical expectation) of the queue length process $X(t)$ at the moment t under the initial condition $X(0) = k$.

Recall that a Markov chain $X(t)$ is called *weakly ergodic*, if $\|\mathbf{p}^*(t) - \mathbf{p}^{**}(t)\| \rightarrow 0$ as $t \rightarrow \infty$ for any initial conditions $\mathbf{p}^*(0), \mathbf{p}^{**}(0)$, where $\mathbf{p}^*(t)$ and $\mathbf{p}^{**}(t)$ are the corresponding solutions of (3). A Markov chain $X(t)$ has the *limiting mean* $\varphi(t)$, if $\lim_{t \rightarrow \infty} (\varphi(t) - E(t, k)) = 0$ for any k .

2 Ergodicity bounds

Consider an increasing sequence of positive numbers $\{d_i\}$, $i = 0, 1, 2, \dots$, $d_0 = 1$, and the corresponding triangular matrix D :

$$D = \begin{pmatrix} d_1 & d_1 & d_1 & \cdots \\ 0 & d_2 & d_2 & \cdots \\ 0 & 0 & d_3 & \cdots \\ & \ddots & \ddots & \ddots \end{pmatrix} \quad (4)$$

Let l_{1D} be the space of sequences:

$$l_{1D} = \{\mathbf{z} = (p_1, p_2, \dots)^T : \|\mathbf{z}\|_{1D} \equiv \|D\mathbf{z}\| < \infty\}.$$

We also introduce the auxiliary space of sequences l_{1E} as

$$l_{1E} = \left\{ \mathbf{z} = (p_1, p_2, \dots)^T : \|\mathbf{z}\|_{1E} \equiv \sum k|p_k| < \infty \right\}.$$

Put

$$W = \inf_{i \geq 1} \frac{d_i}{i}, \quad g_i = \sum_{n=1}^i d_n.$$

Consider the following expressions:

$$\begin{aligned} \alpha_k(t) = \lambda_k(t) + \mu_{k+1}(t) + \xi_{k+1}(t) - \frac{d_{k+1}}{d_k} \lambda_{k+1}(t) - \\ \frac{d_{k-1}}{d_k} \mu_k(t), \quad k \geq 0, \end{aligned} \quad (5)$$

and

$$\alpha(t) = \inf_{k \geq 0} \alpha_k(t). \quad (6)$$

Now recall the following general statement.

Theorem 1 *Let $X(t)$ be a birth-death-catastrophe process (BDPC) with rates $\lambda_k(t)$, $\mu_k(t)$ and $\xi_k(t)$. Assume that there exists a sequence $\{d_i\}$ such that*

$$\int_0^\infty \alpha(t) dt = +\infty. \quad (7)$$

Then $X(t)$ is weakly ergodic, and the following bounds hold:

$$\|\mathbf{p}^*(t) - \mathbf{p}^{**}(t)\|_{1D} \leq e^{-\int_s^t \alpha(\tau) d\tau} \|\mathbf{p}^*(s) - \mathbf{p}^{**}(s)\|_{1D}, \quad (8)$$

$$\|\mathbf{p}^*(t) - \mathbf{p}^{**}(t)\| \leq 4e^{-\int_s^t \alpha(\tau) d\tau} \sum_{i \geq 1} g_i |p_i^*(s) - p_i^{**}(s)|, \quad (9)$$

for any $t \geq s \geq 0$ and any initial conditions $\mathbf{p}^(s), \mathbf{p}^{**}(s)$.*

Proof. The proof follows the lines of the reasoning used to prove Theorem 3 in [29], hence we only outline this argumentation here. Put $p_0(t) = 1 - \sum_{i \geq 1} p_i(t)$, then from (3) we have the following system:

$$\frac{d\mathbf{z}(t)}{dt} = B(t)\mathbf{z}(t) + \mathbf{f}(t), \quad (10)$$

where $\mathbf{z}(t) = (p_1(t), p_2(t), \dots)^T$, $\mathbf{f}(t) = (\lambda_0(t), 0, 0, \dots)^T$, $B(t) = (b_{ij}(t))_{i,j=1}^\infty$ and

$$b_{ij} = \begin{cases} -(\lambda_0 + \lambda_1 + \mu_1 + \xi_1), & \text{if } i = j = 1, \\ \mu_2 - \lambda_0, & \text{if } i = 1, j = 2, \\ -\lambda_0, & \text{if } i = 1, j > 2, \\ -(\lambda_j + \mu_j + \xi_j), & \text{if } i = j > 1, \\ \mu_j, & \text{if } i = j - 1 > 1, \\ \lambda_j, & \text{if } i = j + 1 > 1, \\ 0, & \text{otherwise.} \end{cases} \quad (11)$$

This is a linear non-homogeneous differential system, the solution of which can be written as

$$\mathbf{z}(t) = V(t, 0)\mathbf{z}(0) + \int_0^t V(t, \tau)\mathbf{f}(\tau) d\tau, \quad (12)$$

where $V(t, z)$ is the Cauchy operator of (10), see, for instance, [29].

We can consider (10) as a differential equation in the space l_{1D} with bounded and locally integrable on $[0, \infty)$ coefficients $\mathbf{f}(t)$ and $B(t)$.

Applying the notion of the logarithmic norm and the related bounds (see [14, 15, 20, 29] for details), we obtain the following bound for the logarithmic norm $\gamma(B(t))$ in l_{1D} :

$$\begin{aligned} \gamma(B)_{1D} &= \gamma(DB(t)D^{-1})_1 = \\ \sup_{i \geq 0} \left(\frac{d_{i+1}}{d_i} \lambda_{i+1}(t) + \frac{d_{i-1}}{d_i} \mu_i(t) - (\lambda_i(t) + \mu_{i+1}(t) + \xi_{i+1}(t)) \right) &= \\ - \inf_{k \geq 0} (\alpha_k(t)) &= -\alpha(t), \end{aligned} \quad (13)$$

in accordance with (6). Hence,

$$\|V(t, s)\|_{1D} \leq e^{-\int_s^t \alpha(\tau) d\tau}. \quad (14)$$

Therefore, bound (8) takes place.

On the other hand, inequalities $\|\mathbf{z}\| \leq 2\|\mathbf{z}\|_{1D}$, and $\|\mathbf{p}^* - \mathbf{p}^{**}\| \leq 2\|\mathbf{z}\|$ for any $\mathbf{p}^*, \mathbf{p}^{**}$ and corresponding \mathbf{z} (see, for instance [29]) imply bound (9).

Corollary 1 *Let, in addition, the numbers d_i grow sufficiently fast so that $W > 0$. Then $X(t)$ has the limiting mean, say $\phi(t)$, and the following bound holds:*

$$|\phi(t) - E_k(t)| \leq \frac{4}{W} e^{-\int_0^t \alpha(\tau) d\tau} \|\mathbf{p}(0) - \mathbf{e}_k\|_{1D}. \quad (15)$$

Now we can obtain ergodicity bounds for the queue length process of an $M_t|M_t|S$ queue with catastrophes.

Theorem 2 *Let $\xi_k(t) = \zeta_k \xi(t)$,*

$$\inf_n \zeta_n = \zeta > 0 \quad (16)$$

and let there exist $\varepsilon > 0$ such that

$$\int_0^\infty (\zeta \xi(t) - \varepsilon \lambda(t)) dt = +\infty \quad (17)$$

(large catastrophe rates). Then the queue-length process $X(t)$ is weakly ergodic, has the limiting mean, and the following bounds hold:

$$\|\mathbf{p}(t) - \pi(t)\| \leq 4(1 + \varepsilon)^k \varepsilon^{-1} e^{-\int_0^t (\zeta \xi(\tau) - \varepsilon \lambda(\tau)) d\tau} \quad (18)$$

$$|E_k(t) - E_0(t)| \leq \frac{4(1 + \varepsilon)^k}{\varepsilon W} e^{-\int_0^t (\zeta \xi(\tau) - \varepsilon \lambda(\tau)) d\tau}, \quad (19)$$

for any initial number of customers $X(0) = k$, where $\pi(t)$ and $\phi(t) = E_0(t)$ are the limiting regime and the limiting mean correspondent to the empty initial length of the queue.

Proof. Put $d_0 = 1$, $d_{k+1} = (1 + \varepsilon)d_k$, $k \geq 0$, then instead of (13) we have the following bound for the logarithmic norm:

$$\gamma(B(t))_{1D} \leq - \left(\zeta \xi(t) + \frac{\varepsilon}{1 + \varepsilon} (S\mu(t) - (1 + \varepsilon)\lambda(t)) \right) \leq \quad (20)$$

$$- (\zeta \xi(t) - \varepsilon \lambda(t)) = -\alpha_*(t),$$

where $\int_0^\infty \alpha_*(t) dt = +\infty$ in accordance with (16).

Putting $\mathbf{p}^*(0) = \pi(0) = \mathbf{e}_0$, $\mathbf{p}^{**}(0) = \mathbf{p}(0) = \mathbf{e}_k$, from Theorem 1 we obtain bound (18). The second estimate follows from (15) for $W = \inf_{k \geq 1} \frac{(1+\varepsilon)^k}{k} > 0$.

Theorem 3 *Let there exist $\varepsilon > 0$ such that*

$$\int_0^\infty (S\mu(t) - (1 + \varepsilon)\lambda(t)) dt = +\infty \quad (21)$$

(large service rates). Then the process $X(t)$ is weakly ergodic and has the limiting mean. Moreover, the following bounds hold:

$$\|\mathbf{p}(t) - \pi(t)\| \leq 4(1 + \varepsilon)^k \varepsilon^{-1} e^{-\int_0^t \frac{\varepsilon}{1+\varepsilon} (S\mu(\tau) - (1+\varepsilon)\lambda(\tau)) d\tau} \quad (22)$$

and

$$|E_k(t) - E_0(t)| \leq \frac{4(1 + \varepsilon)^k}{\varepsilon W} e^{-\int_0^t \frac{\varepsilon}{1+\varepsilon} (S\mu(\tau) - (1+\varepsilon)\lambda(\tau)) d\tau}, \quad (23)$$

for any initial number of customers $X(0) = k$, where $\pi(t)$ and $\phi(t) = E_0(t)$ are the limiting regime and the limiting mean corresponding to the empty initial queue.

Proof. Put $d_0 = 1$, $d_{k+1} = (1 + \varepsilon)d_k$, $k \geq 0$, then instead of (13) and (21) we have the following bound of the logarithmic norm:

$$\gamma(B(t))_{1D} \leq - \left(\zeta \xi(t) + \frac{\varepsilon}{1 + \varepsilon} (S\mu(t) - (1 + \varepsilon)\lambda(t)) \right) \leq \quad (24)$$

$$- \frac{\varepsilon}{1 + \varepsilon} (S\mu(t) - (1 + \varepsilon)\lambda(t)) = -\alpha_*(t).$$

This estimate implies our claim.

Remark 1 *Perturbation bounds for general inhomogeneous $M_t|M_t|S$ queue with catastrophes can be formulated using the general approach of [28] and previous bounds on the rate of convergence.*

3 Truncations

Unfortunately, the structure of the infinitesimal matrix of the process does not provide uniform truncation bounds, as in [29]. Instead, we can apply another approach to finding simple and sufficiently sharp truncation bounds, the first such example was considered in [24].

Consider the family of “truncated” processes $X_n(t)$ on the state space $E_n = \{0, 1, \dots, n\}$ with the corresponding reduced intensity matrix $A_n(t)$. Below we will identify the finite vector with entries, say, $(a_1, a_2, \dots, a_n)^T$ and the infinite vector with the same first n coordinates and the others equal to zero. In addition, we suppose that

$$e^{-\int_s^t \alpha(u) du} \leq M e^{-a(t-s)}, \quad (25)$$

for some positive M, a and any s, t , $0 \leq s \leq t$. Put $W_n = \inf_{k \geq n} \frac{\sum_{i=n}^k d_i}{k}$.

Theorem 4 *Let the assumptions of Theorem 1 be fulfilled, and, in addition, let (25) hold. Then*

$$\|\mathbf{p}(t) - \mathbf{p}_n(t)\| \leq \frac{8Lt}{nW_n} \left(Mj d_{j+1} + \frac{LMd_1}{a} \right), \quad (26)$$

$$|E_{\mathbf{p}}(t) - E_{\mathbf{p}_n}(t)| \leq \frac{3L(n+1)t}{nW_n} \left(Mj d_{j+1} + \frac{LMd_1}{a} \right). \quad (27)$$

for any $t \geq 0$, and initial condition $\mathbf{p}(0) = \mathbf{p}_n(0) = \mathbf{e}_j$.

Proof. Consider the forward Kolmogorov equation for $X(t)$ and $X_n(t)$ respectively in the following form:

$$\frac{d\mathbf{p}}{dt} = A_n(t)\mathbf{p} + (A(t) - A_n(t))\mathbf{p}, \quad (28)$$

and

$$\frac{d\mathbf{p}_n}{dt} = A_n(t)\mathbf{p}_n. \quad (29)$$

We have

$$\mathbf{p}_n(t) = U_n(t, 0)\mathbf{p}(0) \quad (30)$$

if $\mathbf{p}(0) = \mathbf{p}_n(0)$ and

$$\mathbf{p}(t) = U_n(t, 0)\mathbf{p}(0) + \int_0^t U_n(t, \tau) (A(\tau) - A_n(\tau))\mathbf{p}(\tau) d\tau. \quad (31)$$

Then in *any* norm we have

$$\|\mathbf{p}(t) - \mathbf{p}_n(t)\| = \left\| \int_0^t U_n(t, \tau) (A(\tau) - A_n(\tau)) \mathbf{p}(\tau) d\tau \right\|. \quad (32)$$

Consider the Cauchy matrix

$$U_n = \begin{pmatrix} u_{00}^n & \cdot & \cdot & u_{0n}^n & 0 & 0 & \cdots \\ u_{10}^n & \cdot & \cdot & u_{1n}^n & 0 & 0 & \cdots \\ \cdots & & & & & & \\ u_{n0}^n & \cdot & \cdot & u_{nn}^n & 0 & 0 & \cdots \\ 0 & \cdot & \cdot & 0 & 1 & 0 & \cdots \\ 0 & \cdot & \cdot & 0 & 0 & 1 & \cdots \\ \cdots & & & & & & \end{pmatrix}. \quad (33)$$

Then

$$(A - A_n) \mathbf{p} =$$

$$= \left(\sum_{i>n} \xi_i(t) p_i, 0, \dots, -\lambda_n(t) p_n + \mu_{n+1}(t) p_{n+1}, \lambda_n(t) p_n - (\lambda_{n+1}(t) + \mu_{n+1}(t) + \xi_{n+1}(t)) p_{n+1} + \mu_{n+2}(t) p_{n+2}, \dots \right)^T$$

and hence

$$U_n (A - A_n) \mathbf{p} = \begin{pmatrix} u_{00}^n \sum_{i>n} \xi_i(t) p_i + u_{0n}^n (-\lambda_n(t) p_n + \mu_{n+1}(t) p_{n+1}) \\ u_{10}^n \sum_{i>n} \xi_i(t) p_i + u_{1n}^n (-\lambda_n(t) p_n + \mu_{n+1}(t) p_{n+1}) \\ \vdots \\ u_{n0}^n \sum_{i>n} \xi_i(t) p_i + u_{nn}^n (-\lambda_n(t) p_n + \mu_{n+1}(t) p_{n+1}) \\ \lambda_n(t) p_n - (\lambda_{n+1}(t) + \mu_{n+1}(t) + \xi_{n+1}(t)) p_{n+1} + \mu_{n+2}(t) p_{n+2} \\ \lambda_{n+1}(t) p_{n+1} - (\lambda_{n+2}(t) + \mu_{n+2}(t) + \xi_{n+2}(t)) p_{n+2} + \mu_{n+3}(t) p_{n+3} \\ \vdots \end{pmatrix}. \quad (34)$$

$$\begin{aligned} \|U_n (A - A_n) \mathbf{p}\| &\leq \\ \sum_{k \geq 0}^n \left| u_{k0}^n \sum_{i>n} \xi_i(t) p_i \right| &+ \sum_{k \geq 0}^n |u_{kn}^n (-\lambda_n(t) p_n + \mu_{n+1}(t) p_{n+1})| + \\ \sum_{k \geq n} |\lambda_k(t) p_k - (\lambda_{k+1}(t) + \mu_{k+1}(t) + \xi_{k+1}(t)) p_{k+1} + \mu_{k+2}(t) p_{k+2}| &\leq \\ L \sum_{i>n} p_i + |\lambda_n(t) p_n| + |\mu_{n+1}(t) p_{n+1}| &+ \\ 2 \sum_{k \geq n} |\lambda_k(t) p_k| + 2 \sum_{k \geq n+1} |\mu_k(t) p_k| + \sum_{k \geq n+1} |\xi_k(t) p_k| &\leq \frac{8L}{n} \sum_{k \geq n} k p_k, \end{aligned} \quad (35)$$

$$\begin{aligned}
& \|U_n(A - A_n)\mathbf{p}\|_{1E} = \\
& \sum_{k \geq 1}^n k \left| u_{k0}^n \sum_{i > n} \xi_i(t) p_i \right| + \sum_{k \geq 1}^n k |u_{kn}^n (-\lambda_n(t) p_n + \mu_{n+1}(t) p_{n+1})| + \\
& \sum_{k \geq n} (k+1) |\lambda_k(t) p_k - (\lambda_{k+1}(t) + \mu_{k+1}(t) + \xi_{k+1}(t)) p_{k+1} + \mu_{k+2}(t) p_{k+2}| \leq \\
& L \sum_{k \geq n} (2k+1) p_k \leq \frac{3L(n+1)}{n} \sum_{k \geq n} k p_k. \quad (36)
\end{aligned}$$

On the other hand,

$$\begin{aligned}
\|\mathbf{p}\|_{1D} & \geq d_n(p_n + p_{n+1} + \dots) + d_{n+1}(p_{n+1} + p_{n+2} + \dots) + \dots = \\
& p_n d_n + p_{n+1}(d_n + d_{n+1}) + \dots = \\
& \frac{d_n}{n} n p_n + \frac{d_n + d_{n+1}}{n+1} p_{n+1} + \dots \geq W_n \sum_{k \geq n} k p_k. \quad (37)
\end{aligned}$$

Therefore we have

$$\begin{aligned}
\|\mathbf{p}\|_{1D} & \leq \|V(t, 0)\mathbf{p}(0)\|_{1D} + \int_0^t \|V(t, \tau)\mathbf{f}(\tau)\|_{1D} d\tau \leq \\
& e^{-\int_0^t \alpha(u) du} \|\mathbf{p}(0)\|_{1D} + \int_0^t \lambda_0(\tau) e^{-\int_\tau^t \alpha(u) du} d\tau \leq \\
& M e^{-at} \|\mathbf{p}(0)\|_{1D} + \frac{LMd_1}{a} \leq M j d_{j+1} + \frac{LMd_1}{a} \quad (38)
\end{aligned}$$

for any $\mathbf{p}(0) = \mathbf{e}_j$, since $\|\mathbf{f}(\tau)\|_{1D} \leq d_1 L$, and bounds (26), (27) hold.

Corollary 2 *Under the assumptions of Theorem 2, let there exist positive M, a such that $e^{-\int_s^t (\xi \xi(u) - \varepsilon \lambda(u)) du} \leq M e^{-a(t-s)}$, for any s, t , $0 \leq s \leq t$ (large catastrophe rates). Then the following bounds hold:*

$$\|\mathbf{p}(t) - \mathbf{p}_n(t)\| \leq \frac{8Lt}{nW_n} \left(M j (1 + \varepsilon)^{j+1} + \frac{LM(1 + \varepsilon)}{a} \right), \quad (39)$$

$$|E_{\mathbf{p}}(t) - E_{\mathbf{p}_n}(t)| \leq \frac{3L(n+1)t}{nW_n} \left(M j (1 + \varepsilon)^{j+1} + \frac{LM(1 + \varepsilon)}{a} \right), \quad (40)$$

for any $t \geq 0$, and any initial condition $\mathbf{p}(0) = \mathbf{p}_n(0) = \mathbf{e}_j$.

Corollary 3 *Under the assumptions of Theorem 3, let there exist M, a such that*

$$e^{-\int_s^t (S\mu(u) - (1+\varepsilon)\lambda(u)) du} \leq Me^{-a(t-s)}, \quad (41)$$

for any s, t , $0 \leq s \leq t$ (large service rates). Then the following bounds hold:

$$\|\mathbf{p}(t) - \mathbf{p}_n(t)\| \leq \frac{8Lt}{nW_n} \left(Mj(1+\varepsilon)^{j+1} + \frac{LM(1+\varepsilon)}{a} \right), \quad (42)$$

and

$$|E_{\mathbf{p}}(t) - E_{\mathbf{p}_n}(t)| \leq \frac{3L(n+1)t}{nW_n} \left(Mj(1+\varepsilon)^{j+1} + \frac{LM(1+\varepsilon)}{a} \right), \quad (43)$$

for any $t \geq 0$, and any initial condition $\mathbf{p}(0) = \mathbf{p}_n(0) = \mathbf{e}_j$.

4 Example

Consider an $M_t|M_t|S$ queue with catastrophes in the case of large S and periodic intensities. Let $S = 10^{12}$, $\lambda(t) = 1 + \sin 2\pi t$, $\mu(t) = 3 + 2 \cos 2\pi t$, $\xi_k(t) = \zeta_k \xi(t)$, where $\xi(t) = 1 - \sin 2\pi t$ and $\zeta_k = 1 + 1/k$. A similar example without catastrophes was considered in [27] and [29].

Here we briefly discuss the way for choosing $\{d_i\}$. Firstly, the monotonicity of this sequence implies the bounds

$$\alpha_0(t) \leq \mu(t) + \xi_1(t), d_1 = 1,$$

hence

$$\alpha_1(t) \leq \mu(t) + \xi_2(t), d_2 = 1,$$

and so on. Therefore, the best possible bound of the “decay function” is

$$\alpha^*(t) = \mu(t) + \inf_{k \geq 0} \xi_k(t) = \mu(t) + \xi(t) = 4 + 2 \cos 2\pi t - \sin 2\pi t.$$

On the other hand, such approach yields small values of W and W_n . Therefore, we obtain bad scores both for the rate of convergence to the limiting mean and for the error of truncations.

Finally we choose the “average” sequence $\{d_i\}$, namely, putting $d_{k+1} = 2^k$ for any $k \geq 0$, we have

$$W = \inf_{i \geq 1} \frac{d_i}{i} = 1, \quad g_k = \sum_{n=1}^k d_n \leq 2^k,$$

$$\alpha_k(t) \geq \mu(t) + \xi(t) - \lambda(t).$$

Therefore, Theorem 1 gives us the weak ergodicity of $X(t)$. Moreover, if the limiting regime and limiting mean correspond to the initial condition $X(0) = 0$, then $\mathbf{p}^{**}(0) = \mathbf{e}_0$ and $\phi(0) = 0$, and the following bounds hold:

$$\|\mathbf{p}^*(t) - \mathbf{p}^{**}(t)\| \leq 2^{k+2} e^{-\int_0^t (3+2 \cos 2\pi\tau - 2 \sin 2\pi\tau) d\tau} \leq 2^{k+4} e^{-3t}, \quad (44)$$

$$|\phi(t) - E_k(t)| \leq 2^{k+2} e^{-\int_0^t (3+2 \cos 2\pi\tau - 2 \sin 2\pi\tau) d\tau} \leq 2^{k+4} e^{-3t}, \quad (45)$$

for any $t \geq s \geq 0$ and any initial condition $X(0) = k$.

Consider the error of truncations. We have $M \leq 4$, $a = 3$, $L \approx 5 \cdot 10^{12}$, and $W_n = \inf_{k \geq n} \frac{\sum_{i=n}^k d_i}{k} = \frac{2^{n-1}}{n}$. Hence, the following bounds follow from Theorem 4:

$$\|\mathbf{p}(t) - \mathbf{p}_n(t)\| \leq \frac{t \cdot 10^{13}}{2^{n-3}} (k2^{k+2} + 10^{14}), \quad (46)$$

$$|E_{\mathbf{p}}(t) - E_{\mathbf{p}_n}(t)| \leq \frac{t(n+1) \cdot 10^{14}}{2^{n-1}} (k2^{k+2} + 10^{14}), \quad (47)$$

for any $t \geq 0$, and any initial condition $X(0) = k$.

Therefore, we can choose $n = 120$, $t \in [6, 7]$ and find the limiting characteristics with error less than 10^{-6} .

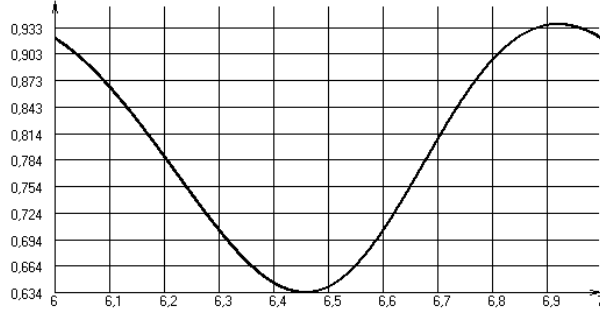


Figure 1: Approximation of the limiting probability of empty queue $\Pr\{X(t) = 0 | X(0) = 0\}$ on $[6, 7]$.

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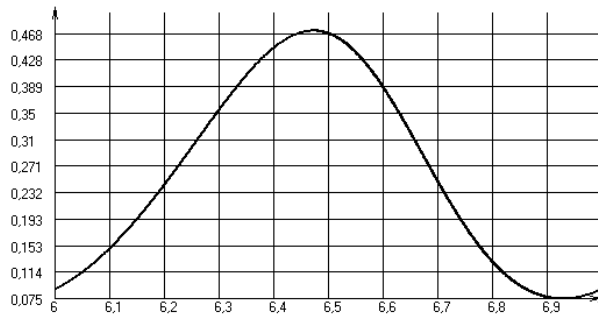


Figure 2: Approximation of the limiting mean $E(t, 0)$ on $[6, 7]$.

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